

# CONSTRAINED LATTICE-FIELD HIERARCHIES AND TODA SYSTEM WITH BLOCK SYMMETRY

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**ABSTRACT.** In this paper, we construct the additional  $W$ -symmetry and ghost symmetry of two-lattice field integrable hierarchies. Using the symmetry constraint, we construct constrained two-lattice integrable systems which contain several new integrable difference equations. Under a further reduction, the constrained two-lattice integrable systems can be combined into one single integrable system, namely the well-known one dimensional original Toda hierarchy. We prove that the one dimensional original Toda hierarchy has a nice Block Lie symmetry.

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## 1. INTRODUCTION

The most fundamental integrable models in mathematical physics are the KP system and Toda system. As one of the most important sub-hierarchies of the KP hierarchy by considering reductions on the Lax operator, the constrained KP hierarchy contains a large number of interesting soliton equations under the so-called symmetry constraint[1, 2, 3]. One of the constraints means that the negative part of the Lax operator of the constrained KP hierarchy is a generator of the additional symmetries of the KP hierarchy. The Toda hierarchy is a completely integrable system which has many important applications in mathematics and physics including the theory of Lie algebras and so on [4, 5]. The Toda system has many kinds of reductions or extensions, for example extended Toda hierarchy[6], bigraded Toda hierarchy [7]-[10] which governs the Gromov-Witten invariant of  $CP^1$  and orbifolds. In this paper, we will construct two constrained lattice-field hierarchies which are similar to two dimensional Toda system but without crossing flows.

Additional symmetries of the KP hierarchy were given by Orlov and Shulman [11] through the Orlov-Shulman operator  $M$ , which can be used to form a centerless  $W$  algebra. The generating function of additional  $W$ -symmetries of the KP type integrable hierarchy constitutes a squared eigenfunction symmetry or ghost symmetry in terms of wave functions[12, 13, 14]. Because of the universality of additional symmetries for integrable systems, in this paper, we will use the additional symmetry of two-lattice field hierarchies to do a further reduction. The reduction

produces constrained lattice-field hierarchies whose further reduction is the well-known original Toda hierarchy. The infinite dimensional Lie algebra of Block type is a generalization of the well-known Virasoro algebra and has been studied intensively for example in literatures [15, 16]. Later we provide this kind of Block type algebraic structure for the bigraded Toda hierarchy [10], dispersionless bigraded Toda hierarchy [17] and D type Drinfeld-Sokolov hierarchy [18]. In this paper, we will further prove that the reduced Toda hierarchy recovers the original Toda hierarchy with its additional Block Lie algebra.

## 2. TWO-LATTICE FIELD HIERARCHIES

Two-lattice field hierarchies considered in this section are two families of evolution equations depending on infinitely many variables  $x = (x_1, x_2, \dots)$ ,  $y = (y_1, y_2, \dots)$  respectively and a difference variable  $n$ . Basing on the discrete KP hierarchy in [22] and Toda hierarchy in [4, 5], now we consider two-lattice field hierarchies as

$$\frac{\partial L}{\partial x_i} = [B_i, L], \quad B_i := (L^i)_+, \quad (2.1)$$

$$\frac{\partial \bar{L}}{\partial y_i} = [\bar{B}_i, \bar{L}], \quad \bar{B}_i := (\bar{L}^i)_+, \quad (2.2)$$

where  $L, \bar{L}$  are two general pseudo-shift operators

$$L(n) = \Lambda + \sum_{j=0}^{\infty} u_j(n) \Lambda^{-j}, \quad (2.3)$$

$$\bar{L}(n) = v_{-1} \Lambda^{-1} + \sum_{j=0}^{\infty} v_j(n) \Lambda^j. \quad (2.4)$$

These two-lattice field hierarchies can be treated as one part of the two dimensional Toda lattice hierarchy [5] without crossing flow equations. Here the shift operator  $\Lambda$  acts on a discrete function  $f(n)$  as  $\Lambda f(n) = f(n+1)$ . Similar to the two dimensional Toda hierarchy [5],  $L$  and  $\bar{L}$  can also be dressed by dressing operators  $\mathcal{P}$  and  $\bar{\mathcal{P}}$

$$\mathcal{P}(n; x) = 1 + \sum_{j=1}^{\infty} a_j(n; x) \Lambda^{-j}, \quad (2.5)$$

$$\bar{\mathcal{P}}(n; y) = \bar{a}_0 + \sum_{j=1}^{\infty} \bar{a}_j(n; y) \Lambda^j, \quad (2.6)$$

by

$$L = \mathcal{P} \circ \Lambda \circ \mathcal{P}^{-1}, \quad \bar{L} = \bar{\mathcal{P}} \circ \Lambda^{-1} \circ \bar{\mathcal{P}}^{-1}. \quad (2.7)$$

Define an anti-evolution operator  $*$  by:  $\Lambda^* = \Lambda^{-1}$ ,  $f(n)^* = f(n)$  for an arbitrary function  $f(n)$ . Also we define two wave functions (adjoint wave functions)  $w, \bar{w}(w^*, \bar{w}^*)$  as following

$$w = \mathcal{P} e^{\xi(n, x, z)}, \quad w^* = \mathcal{P}^{-1*} e^{-\xi(n, x, z)}, \quad (2.8)$$

$$\bar{w} = \bar{\mathcal{P}} e^{\bar{\xi}(n,y,z)}, \quad \bar{w}^* = \bar{\mathcal{P}}^{-1*} e^{-\bar{\xi}(n,y,z)}, \quad (2.9)$$

where,

$$\xi(n, x, z) = \sum_{m \geq 0} z^m x_m + n \ln z, \quad \bar{\xi}(n, y, z) = - \sum_{m \geq 0} z^{-m} y_m + n \ln z. \quad (2.10)$$

One can find the wave functions  $w, \bar{w}$  and their adjoint wave functions  $w^*, \bar{w}^*$  satisfy the following flow equations

$$\frac{\partial w}{\partial x_i} = L_+^n w, \quad \frac{\partial w^*}{\partial x_i} = -L_+^{n*} w^*, \quad (2.11)$$

$$\frac{\partial \bar{w}}{\partial y_i} = -\bar{L}_-^n \bar{w}, \quad \frac{\partial \bar{w}^*}{\partial y_i} = \bar{L}_-^{n*} \bar{w}^*. \quad (2.12)$$

Based on the above dressing structures in eq.(2.7), two independent additional symmetries will be given in the next section which will be used to construct the constrained system by symmetry constraints.

### 3. $W$ -SYMMETRY AND GHOST SYMMETRY

In order to give the additional symmetries of two-lattice field hierarchies, similarly as [21], we can define the Orlov-Schulman's  $M_L, M_R$  operators by

$$M_L = \mathcal{P} \Gamma_L \mathcal{P}^{-1}, \quad M_R = \bar{\mathcal{P}} \Gamma_R \bar{\mathcal{P}}^{-1}, \quad (3.1)$$

where

$$\Gamma_L = n\Lambda^{-1} + \sum_{i \geq 0} (i+1)\Lambda^i x_i, \quad \Gamma_R = -n\Lambda - \sum_{i \geq 0} (i+1)\Lambda^{-i} y_i. \quad (3.2)$$

Like the additional symmetry of the Toda hierarchy in [21], we are now to recall the definition of the additional flows, and then to prove that they are symmetries, which are called additional symmetries of two-lattice field hierarchies. The additional flows over two sets of independent variables  $t_{m,l}^*, \bar{t}_{m,l}^*$  and their actions on the wave operators are defined as

$$\frac{\partial \mathcal{P}}{\partial t_{m,l}^*} = - (M_L^m L^l)_- \mathcal{P}, \quad \frac{\partial \bar{\mathcal{P}}}{\partial \bar{t}_{m,l}^*} = (M_R^m \bar{L}^l)_+ \bar{\mathcal{P}}, \quad (3.3)$$

where  $m \geq 0, l \geq 0$ . The additional flows can be proved to commute with the flows of the two-lattice field hierarchies, i.e.,

$$[\partial_{t_{m,l}^*}, \partial_{x_n}] \Phi = 0, \quad [\partial_{\bar{t}_{m,l}^*}, \partial_{y_n}] \Psi = 0, \quad (3.4)$$

where  $\Phi$  can be  $\mathcal{P}, L$ , and  $\Psi$  can be  $\bar{\mathcal{P}}$  or  $\bar{L}$ , and  $\partial_{t_{m,l}^*} = \frac{\partial}{\partial t_{m,l}^*}, \partial_{x_n} = \frac{\partial}{\partial x_n}, \partial_{y_n} = \frac{\partial}{\partial y_n}$ .

The commutative property means that additional flows are symmetries of the lattice-field hierarchy. Since they are symmetries, one can find that the algebraic structures among these additional symmetries are two separated  $W$ -algebras which is included in the following known important proposition.

**Proposition 3.1.** *The additional flows  $\partial_{t_{m,l}^*}$  form a  $W$ -algebras with the following relation*

$$[\partial_{t_{m,l}^*}, \partial_{t_{n,k}^*}] \mathcal{P} = C_{m,l,n,k}^{p,q} \partial_{t_{p,q}^*} \mathcal{P}, \quad [\partial_{\bar{t}_{m,l}^*}, \partial_{\bar{t}_{n,k}^*}] \bar{\mathcal{P}} = C_{m,l,n,k}^{p,q} \partial_{\bar{t}_{p,q}^*} \bar{\mathcal{P}}, \quad (3.5)$$

where  $C_{m,l,n,k}^{p,q}$  is the coefficients of the  $W$ -algebra and  $m, n, l, k \geq 0$ .

Similar as the method in [21], two generating functions of independent additional symmetries can be constructed as following

$$Y(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} (M_L^m L^{l+m})_-, \quad (3.6)$$

$$\bar{Y}(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} (M_R^m L^{l+m})_+, \quad (3.7)$$

which can be expressed by a simple form in the sequent proposition. To this end, we need several well known and useful techniques in the following several lemmas. Here we define  $\text{res}_\lambda \sum_{i=-\infty}^{+\infty} A_i \lambda^i = A_{-1}$ ,  $\text{res}_\Lambda \sum_{i=-\infty}^{+\infty} A_i \Lambda^i = A_0$ , which will be used in the following lemma.

**Lemma 3.2.** *For two pseudo-shift operators  $P$  and  $Q$ , the identities*

$$\text{res}_z[(P e^{\xi(n,x,z)})(Q(n-1)e^{-\xi(n,x,z)})] = \text{res}_\Lambda[PQ^*], \quad (3.8)$$

$$\text{res}_z[(P e^{\bar{\xi}(n,y,z)})(Q(n-1)e^{-\bar{\xi}(n,y,z)})] = \text{res}_\Lambda[PQ^*] \quad (3.9)$$

hold true.

*Proof.* We suppose

$$P = \sum_{i=-\infty}^{\infty} p_i \Lambda^i, \quad Q = \sum_{j=-\infty}^{\infty} q_j \Lambda^j. \quad (3.10)$$

Then we get

$$\begin{aligned} & \text{res}_z[(P e^{\xi(n,x,z)})(Q(n-1)e^{-\xi(n,x,z)})] \\ &= \text{res}_z[(\sum_{i=-\infty}^{\infty} p_i \Lambda^i e^{\xi(n,x,z)})(\sum_{j=-\infty}^{\infty} q_j(n-1) \Lambda^j e^{-\xi(n,x,z)})] \\ &= \text{res}_z[\sum_{i=-\infty}^{\infty} p_i z^i \sum_{j=-\infty}^{\infty} q_j(n-1) z^{-j}] \\ &= \sum_{i-j=-1} p_i(n) q_j(n-1), \end{aligned}$$

and

$$\text{res}_\Lambda[PQ^*]$$

$$\begin{aligned}
 &= \text{res}_\Lambda \left[ \sum_{i=-\infty}^{\infty} p_i \Lambda^i \left( \sum_{j=-\infty}^{\infty} q_j \Lambda^j \right)^* \right] \\
 &= \text{res}_\Lambda \left[ \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} p_i \Lambda^i \Lambda^{-j} q_j \right] \\
 &= \sum_{i-j=-1} p_i(n) q_j(n-1),
 \end{aligned}$$

which will finish the proof of the eq.(3.8). For the proof of the eq.(3.9), one can derive it by similar calculations. □

After this, the following two lemmas on residues can also be frequently used.

**Lemma 3.3.** *If  $f(z) = \sum_{-\infty}^{\infty} a_i z^{-i}$ , then*

$$\text{res}_z [\zeta^{-1}(1 - z/\zeta)^{-1} + z^{-1}(1 - \zeta/z)^{-1}] f(z) = f(\zeta). \quad (3.11)$$

(Here  $(1 - z/\zeta)^{-1}$  is understood as a series in  $\zeta^{-1}$  while  $(1 - \zeta/z)^{-1}$  is a series in  $z^{-1}$ .)

**Lemma 3.4.** *Let  $P$  be a pseudo-shift operators  $P = \sum p_i \Lambda^i$ , then*

$$P_- = \sum_{i=1}^{\infty} \Lambda^{-i+1} [\text{res}_\Lambda (\Lambda^{i-1} P)] \Lambda^{-1}, \quad P_+ = \sum_{i=0}^{\infty} \Lambda^{i+1} [\text{res}_\Lambda (\Lambda^{-i-1} P)] \Lambda^{-1}. \quad (3.12)$$

*Proof.* The proof can be derived by the following two direct calculations

$$\begin{aligned}
 P_- &= \sum_{i=1}^{\infty} p_{-i} \Lambda^{-i} \\
 &= \sum_{i=1}^{\infty} \Lambda^{-i+1} p_{-i} (n + (i-1)) \Lambda^{-1} \\
 &= \sum_{i=1}^{\infty} \Lambda^{-i+1} [\text{res}_\Lambda (\Lambda^{i-1} \sum_j p_j \Lambda^j)] \Lambda^{-1} \\
 &= \sum_{i=1}^{\infty} \Lambda^{-i+1} [\text{res}_\Lambda (\Lambda^{i-1} P)] \Lambda^{-1}, \\
 P_+ &= \sum_{i=0}^{\infty} p_i \Lambda^i \\
 &= \sum_{i=0}^{\infty} \Lambda^{i+1} p_i (n - i - 1) \Lambda^{-1}
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{\infty} \Lambda^{i+1} [\text{res}_{\Lambda} (\Lambda^{-i-1} \sum_j p_j \Lambda^j)] \Lambda^{-1} \\
&= \sum_{i=0}^{\infty} \Lambda^{i+1} [\text{res}_{\Lambda} (\Lambda^{-i-1} P)] \Lambda^{-1}.
\end{aligned}$$

□

Then basing on above three lemmas, we can get the following important theorem.

**Theorem 3.5.** *The generating function of additional symmetries  $Y(\lambda, \mu)$ ,  $\bar{Y}(\lambda, \mu)$  have the following simple form*

$$Y(\lambda, \mu) = w(n, x, \mu) \frac{\Lambda^{-1}}{1 - \Lambda^{-1}} w^*(n, x, \lambda), \quad (3.13)$$

$$\bar{Y}(\lambda, \mu) = \bar{w}(n, y, \mu) \frac{1}{1 - \Lambda} \bar{w}^*(n, y, \lambda). \quad (3.14)$$

*Proof.* Using the above three lemmas, we get

$$\begin{aligned}
(M_L^m L^{l+m})_- &= \sum_{i=1}^{\infty} \Lambda^{-i+1} \text{res}_{\Lambda} [\Lambda^{i-1} \mathcal{P} \Gamma_L^m \Lambda^{N(l+m)} \mathcal{P}^{-1}] \Lambda^{-1} \\
&= \sum_{i=1}^{\infty} \Lambda^{-i+1} \text{res}_{\Lambda} [(\Lambda^{i-1} \mathcal{P} \Gamma_L^m \Lambda^{l+m} e^{\xi(n, x, \lambda)}) ((\mathcal{P}^{-1})^*(n-1) e^{-\xi(n, x, \lambda)})] \Lambda^{-1} \\
&= \sum_{i=1}^{\infty} \Lambda^{-i+1} \text{res}_{\Lambda} [(\lambda^{l+m} \Lambda^{i-1} \mathcal{P} \Gamma_L^m e^{\xi(n, x, \lambda)}) (w^*(n-1, x, \lambda))] \Lambda^{-1} \\
&= \sum_{i=1}^{\infty} \Lambda^{-i+1} \text{res}_{\Lambda} [(\lambda^{l+m} \Lambda^{i-1} M_L^m \mathcal{P} e^{\xi(n, x, \lambda)}) (\Lambda^{-1} w^*(n, x, \lambda))] \Lambda^{-1} \\
&= \sum_{i=1}^{\infty} \Lambda^{-i+1} \text{res}_{\Lambda} [(\lambda^{l+m} \Lambda^{i-1} M_L^m w(x, \lambda)) (\Lambda^{-1} w^*(n, x, \lambda))] \Lambda^{-1} \\
&= \sum_{i=1}^{\infty} \Lambda^{-i+1} \text{res}_{\Lambda} [(\lambda^{l+m} (\partial_{\lambda}^m w(x, \lambda))(n + (i-1), x, \lambda)) (w^*(n-1, x, \lambda))] \Lambda^{-1} \\
&= \sum_{i=1}^{\infty} \text{res}_{\Lambda} [(\lambda^{l+m} (\partial_{\lambda}^m w(x, \lambda))(n, x, \lambda)) (w^*(n-i, x, \lambda))] \Lambda^{-i}.
\end{aligned}$$

Take the above result back into  $Y(\lambda, \mu)$ , which becomes

$$Y(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} (M_L^m L^{l+m})_-$$

$$\begin{aligned}
 &= \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} \sum_{i=1}^{\infty} \text{res}_z[(z^{l+m}(\partial_z^m w(x, z)))(w^*(n-i, x, z))]\Lambda^{-i} \\
 &= \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} \sum_{i=1}^{\infty} \text{res}_z[(z^{l+m}(w(n, x, z + \mu - \lambda)))(w^*(n-i, x, z))]\Lambda^{-i} \\
 &= \sum_{l=-\infty}^{\infty} \frac{z^{l+m}}{\lambda^{l+m+1}} \sum_{i=1}^{\infty} \text{res}_z[(w(n, x, z + \mu - \lambda))(w^*(n-i, x, z))]\Lambda^{-i} \\
 &= \sum_{i=1}^{\infty} \text{res}_z[(\frac{1}{z(1-\frac{\lambda}{z})} + \frac{1}{\lambda(1-\frac{z}{\lambda})})w(n, x, z + \mu - \lambda)w^*(n-i, x, z)]\Lambda^{-i} \\
 &= \sum_{i=1}^{\infty} w(n, x, \mu)w^*(n-i, x, \lambda)\Lambda^{-i} \\
 &= \sum_{i=1}^{\infty} w(n, x, \mu)\Lambda^{-i}w^*(n, x, \lambda) \\
 &= w(n, x, \mu)\frac{\Lambda^{-1}}{1-\Lambda^{-1}}w^*(n, x, \lambda).
 \end{aligned}$$

Similarly for the other lattice-field hierarchy, the similar calculations can be done as

$$\begin{aligned}
 (M_R^m L^{l+m})_+ &= \sum_{i=0}^{\infty} \Lambda^{i+1} \text{res}_{\Lambda}[\Lambda^{-i-1} \bar{\mathcal{P}} \Gamma_R^m \Lambda^{-(l+m)} \bar{\mathcal{P}}^{-1}]\Lambda^{-1} \\
 &= \sum_{i=0}^{\infty} \Lambda^{i+1} \text{res}_{\lambda}[(\Lambda^{-i-1} \bar{\mathcal{P}} \Gamma_R^m \Lambda^{-(l+m)} e^{\bar{\xi}(n, y, \lambda)})((\bar{\mathcal{P}}^{-1})^* e^{\bar{\xi}(n, y, \lambda)})]\Lambda^{-1} \\
 &= \sum_{i=0}^{\infty} \Lambda^{i+1} \text{res}_{\lambda}[(\lambda^{-l-m} \Lambda^{-i-1} \bar{\mathcal{P}} \Gamma_R^m e^{-\bar{\xi}(n, y, \lambda)})(\bar{w}^*(n-1, y, \lambda))]\Lambda^{-1} \\
 &= \sum_{i=0}^{\infty} \Lambda^{i+1} \text{res}_{\lambda}[(\lambda^{-l-m} \Lambda^{-i-1} M_R^m \bar{\mathcal{P}} e^{\bar{\xi}(n, y, \lambda)})](\bar{w}^*(n-1, y, \lambda))\Lambda^{-1} \\
 &= \sum_{i=0}^{\infty} \Lambda^{i+1} \text{res}_{\lambda}[(\lambda^{-l-m}(\partial_{\lambda^{-1}}^m \bar{w}(n-i-1, y, \lambda)))(\bar{w}^*(n-1, y, \lambda))]\Lambda^{-1} \\
 &= \sum_{i=0}^{\infty} \text{res}_{\lambda}[(\lambda^{-l-m}(\partial_{\lambda^{-1}}^m \bar{w}(y, \lambda))(n, y, \lambda))(\bar{w}^*(n+i, y, \lambda))]\Lambda^i.
 \end{aligned}$$

Take this back into  $\bar{Y}(\lambda, \mu)$  will lead to

$$\begin{aligned}
\bar{Y}(\lambda, \mu) &= \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{l+m-1} (M_R^m L^{l+m})_+ \\
&= \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{l+m-1} \sum_{i=0}^{\infty} \text{res}_z [(z^{-l-m} (\partial_z^m \bar{w}(y, z))) (\bar{w}^*(n+i, y, z))] \Lambda^i \\
&= \sum_{l=-\infty}^{\infty} \lambda^{l+m-1} \sum_{i=0}^{\infty} \text{res}_z [(z^{-l-m} (\bar{w}(n, y, z + \mu - \lambda))) (\bar{w}^*(n+i, y, z))] \Lambda^i \\
&= \sum_{l=-\infty}^{\infty} \frac{\lambda^{l+m-1}}{z^{l+m}} \sum_{i=0}^{\infty} \text{res}_z [((\bar{w}(n, y, z + \mu - \lambda))) (\bar{w}^*(n+i, y, z))] \Lambda^i \\
&= \sum_{i=0}^{\infty} \text{res}_z [(\frac{1}{z(1-\frac{\lambda}{z})} + \frac{1}{\lambda(1-\frac{z}{\lambda})}) \bar{w}(n, y, z + \mu - \lambda) \bar{w}^*(n+i, y, z)] \Lambda^i \\
&= \sum_{i=0}^{\infty} \bar{w}(n, y, \mu) \bar{w}^*(n+i, y, \lambda) \Lambda^i \\
&= \sum_{i=0}^{\infty} \bar{w}(n, y, \mu) \Lambda^i \bar{w}^*(n, y, \lambda) \\
&= \bar{w}(n, y, \mu) \frac{1}{1-\Lambda} \bar{w}^*(n, y, \lambda),
\end{aligned}$$

where the eq.(3.11) is used. □

The generating functions tell us the lattice-field hierarchies have the following symmetry which is sometimes called “ghost symmetry”.

**Proposition 3.6.** *The two-lattice field hierarchies have the following additional ghost symmetry*

$$\partial_t \mathcal{P}(n, x) = -(w(n, x) \frac{\Lambda^{-1}}{1-\Lambda^{-1}} w^*(n, x)) \mathcal{P}(n, x), \quad (3.15)$$

$$\partial_t \bar{\mathcal{P}}(n, y) = (\bar{w}(n, y) \frac{1}{1-\Lambda} \bar{w}^*(n, y)) \bar{\mathcal{P}}(n, y). \quad (3.16)$$

*Proof.* One can prove the commutativity  $[\partial_{x_n}, \partial_t] \mathcal{P}(n, x) = [\partial_{y_n}, \partial_t] \bar{\mathcal{P}}(n, y) = 0$ , because  $Y(\lambda, \mu)$  and  $\bar{Y}(\lambda, \mu)$  are the generating functions of additional symmetries of two independent lattice-field hierarchies. □



Then by acting on  $e^\xi$  and  $e^{\bar{\xi}}$  respectively one can derive the following flow equations about wave functions  $w, w^*, \bar{w}, \bar{w}^*$ ,

$$\begin{cases} \partial_t w(n, x) = -(w(n, x) \frac{\Lambda^{-1}}{1 - \Lambda^{-1}} w^*(n, x)) w(n, x), \\ \partial_t w^*(n, x) = (w^*(n, x) \frac{\Lambda}{1 - \Lambda} w(n, x)) w(n, x), \end{cases} \quad (3.17)$$

$$\begin{cases} \partial_{\bar{t}} \bar{w}(n, y) = (\bar{w}(n, y) \frac{1}{1 - \Lambda} \bar{w}^*(n, y)) \bar{w}(n, y), \\ \partial_{\bar{t}} \bar{w}^*(n, y) = -(\bar{w}^*(n, y) \frac{1}{1 - \Lambda^{-1}} \bar{w}(n, y)) \bar{w}^*(n, y). \end{cases} \quad (3.18)$$

The eq.(3.17) and eq.(3.18) are two new integrable coupled nonlocal difference equations to our best knowledge.

#### 4. CONSTRAINED TWO-LATTICE FIELD HIERARCHIES

Using the above ghost symmetry, we can do a reduction over the Lax operators of two-lattice field hierarchies, i.e. the Lax operators of the constrained lattice-field hierarchies can be reduced into the following operators

$$\mathcal{L} = \Lambda + u + w(n, x) \frac{\Lambda^{-1}}{1 - \Lambda^{-1}} w^*(n, x), \quad (4.1)$$

$$\bar{\mathcal{L}} = v \Lambda^{-1} + \bar{w}(n, y) \frac{1}{1 - \Lambda} \bar{w}^*(n, y). \quad (4.2)$$

Then the wave function under the above constraints eq.(4.1) and eq.(4.2) will compatible with the following Sato equations

$$\partial_{x_n} w = \mathcal{L}_+^n w, \quad \partial_{x_n} w^* = -(\mathcal{L}_+^n)^* w^*, \quad (4.3)$$

$$\partial_{y_n} \bar{w} = -\bar{\mathcal{L}}_-^n \bar{w}, \quad \partial_{y_n} \bar{w}^* = (\bar{\mathcal{L}}_-^n)^* \bar{w}^*. \quad (4.4)$$

The two constrained lattice-field hierarchies can also be defined as

$$\partial_{x_n} \mathcal{L} = [\mathcal{L}_+^n, \mathcal{L}], \quad \partial_{y_n} \bar{\mathcal{L}} = [-\bar{\mathcal{L}}_-^n, \bar{\mathcal{L}}]. \quad (4.5)$$

Later we will prove that the Lax equations in the eq.(4.5) is compatible with two Lax operators (4.1) and (4.2) using the following proposition.

**Proposition 4.1.** *An operator  $B := \sum_{n=0}^{\infty} b_n \Lambda^n$  is a non-negative shift operator, an operator  $C := \sum_{n=1}^{\infty} c_n \Lambda^{-n}$  is a negative shift operator and  $f(n), g(n)$  are two functions of discrete parameter  $n$ . The following identities hold*

$$(B f \frac{\Lambda^{-1}}{1 - \Lambda^{-1}} g)_- = B(f) \frac{\Lambda^{-1}}{1 - \Lambda^{-1}} g, \quad (f \frac{\Lambda^{-1}}{1 - \Lambda^{-1}} g B)_- = f \frac{\Lambda^{-1}}{1 - \Lambda^{-1}} B^*(g), \quad (4.6)$$

$$(Cf\frac{1}{1-\Lambda}g)_+ = C(f)\frac{1}{1-\Lambda}g, \quad (f\frac{1}{1-\Lambda}gC)_+ = f\frac{1}{1-\Lambda}C^*(g). \quad (4.7)$$

Basing on the Proposition 4.1, two constrained Lax operators (4.1) and (4.2) of the two-lattice field constrained hierarchies are compatible with its corresponding Lax equations. This is included in the following proposition.

**Proposition 4.2.** *The Lax equations in the eq.(4.5) of the two-lattice field constrained hierarchies are compatible with the reduction on two Lax equations (4.1) and (4.2).*

*Proof.* Using Sato equations (2.11) and (2.12), the following calculations directly lead to the compatibility with the projections of both sides of eqs.(4.5)

$$\partial_{x_n}\mathcal{L}(n, x)_- = \partial_{x_n}(w(n, x)\frac{\Lambda^{-1}}{1-\Lambda^{-1}}w^*(n, x)) \quad (4.8)$$

$$= (B_n w(n, x))\frac{\Lambda^{-1}}{1-\Lambda^{-1}}w^*(n, x) \quad (4.9)$$

$$-w(n, x)\frac{\Lambda^{-1}}{1-\Lambda^{-1}}(B_n^* w^*(n, x)) \quad (4.10)$$

$$= [B_n, w(n, x)\frac{\Lambda^{-1}}{1-\Lambda^{-1}}w^*(n, x)]_- \quad (4.11)$$

$$= [B_n, \mathcal{L}]_-, \quad (4.12)$$

$$\partial_{y_n}\bar{\mathcal{L}}(n, y)_+ = \partial_{y_n}(\bar{w}(n, y)\frac{1}{1-\Lambda}\bar{w}^*(n, y)) \quad (4.13)$$

$$= (-\bar{\mathcal{L}}_-^n \bar{w}(n, y))\frac{1}{1-\Lambda}\bar{w}^*(n, y) \quad (4.14)$$

$$+\bar{w}(n, y)\frac{1}{1-\Lambda}(\bar{\mathcal{L}}_-^{n*} \bar{w}^*(n, y)) \quad (4.15)$$

$$= [-\bar{\mathcal{L}}_-^n, \bar{w}(n, y)\frac{1}{1-\Lambda}\bar{w}^*(n, y)]_+ \quad (4.16)$$

$$= [-\bar{\mathcal{L}}_-^n, \bar{\mathcal{L}}]_+. \quad (4.17)$$

Then the compatibility can be seen easily.  $\square$

Comparing with the two dimensional Toda lattice hierarchy, we need to do the following remark on the lattice-field equations.

**Remark 4.3.** *There is an obstacle to define the corresponding constrained two dimensional Toda lattice hierarchy by symmetry constraints eq.(4.1) and eq.(4.2) because one can not define the crossing derivatives of  $x_n$  and  $y_n$  because of a contradiction with Sato equations. In another word, there does not exist a corresponding constrained two-dimensional Toda hierarchy defined by symmetry constraints eq.(4.1) and eq.(4.2) similarly as the constrained KP hierarchy [2, 3].*

Basing on the symmetry constraint of the hierarchy, we choose the first constrained flow equations in equations (4.3) and (4.4) as an example here.

**4.1. Constrained lattice-field equations.** The  $x_1, y_1$  flow equations in equations (4.3) and (4.4) are the following three-components coupled systems

$$\begin{cases} \partial_{x_1} w(n) = w(n+1) + u(n)w(n), \\ \partial_{x_1} w^*(n) = -w^*(n-1) - u(n)w^*(n), \\ \partial_{x_1} u(n) = w(n+1)w^*(n) - w(n)w^*(n-1), \end{cases} \quad (4.18)$$

$$\begin{cases} \partial_{y_1} \bar{w}(n) = v(n)\bar{w}(n-1), \\ \partial_{y_1} \bar{w}^*(n) = -v(n+1)\bar{w}^*(n+1), \\ \partial_{y_1} v(n) = v(n)(\bar{w}(n)\bar{w}^*(n) - \bar{w}(n-1)\bar{w}^*(n-1)). \end{cases} \quad (4.19)$$

The  $x_2, y_2$  flow equations are as

$$\begin{cases} \partial_{x_2} w(n) = [\Lambda^2 + (u(n) + u(n+1))\Lambda + u^2(n) + w(n)w^*(n-1) + w(n+1)w^*(n)]w(n), \\ \partial_{x_2} w^*(n) = -[\Lambda^{-2} + \Lambda^{-1}(u(n) + u(n+1)) + u^2(n) + w(n)w^*(n-1) + w(n+1)w^*(n)]w^*(n), \\ \partial_{x_2} u(n) = (\Lambda^2 - 1)w(n)w^*(n-2) + (\Lambda - 1)(u(n-1) + u(n))(w(n)w^*(n-1)), \\ \partial_{y_2} \bar{w}(n) = -[v(n)v(n-1)\Lambda^{-2} + v(n)(\bar{w}(n-1)\bar{w}^*(n-1) + \bar{w}(n)\bar{w}^*(n))\Lambda^{-1}]\bar{w}(n), \\ \partial_{y_2} \bar{w}^*(n) = [(\Lambda^2 v(n)v(n-1) + \Lambda v(n)(\bar{w}(n-1)\bar{w}^*(n-1) + \bar{w}(n)\bar{w}^*(n)))]\bar{w}^*(n), \\ \partial_{y_2} v(n) = [\bar{w}(n)\bar{w}^*(n+1)v(n+1)v(n) - v(n)v(n-1)\Lambda^{-2}\bar{w}(n)\bar{w}^*(n+1) \\ + v(n)(\bar{w}(n-1)\bar{w}^*(n-1) + \bar{w}(n)\bar{w}^*(n))(\bar{w}(n)\bar{w}^*(n) - \bar{w}(n-1)\bar{w}^*(n-1))]v(n). \end{cases}$$

After denoting  $w(n), w^*(n), u(n), \bar{w}(n), \bar{w}^*(n), v(n)$  as  $q_n, q_n^*, u_n, r_n, r_n^*, v_n$ , then the  $x_1, y_1$  flow equations are as the following simplified form

$$\begin{cases} \partial_{x_1} q_n = q_{n+1} + u_n q_n, \\ \partial_{x_1} q_n^* = -q_{n-1}^* - u_n q_n^*, \\ \partial_{x_1} u_n = q_{n+1} q_n^* - q_n q_{n-1}^*, \end{cases} \quad (4.20)$$

$$\begin{cases} \partial_{y_1} r_n = v_n r_{n-1}, \\ \partial_{y_1} r_n^* = -v_{n+1} r_{n+1}^*, \\ \partial_{y_1} v_n = v_n (r_n r_n^* - r_{n-1} r_{n-1}^*). \end{cases} \quad (4.21)$$

The  $x_2, y_2$  flow equations are as the following simplified form

$$\begin{cases} \partial_{x_2} q_n = q_{n+2} + (u_n + u_{n+1})q_{n+1} + u_n^2 q_n + q_n^2 q_{n-1}^* + q_{n+1} q_n q_n^*, \\ \partial_{x_2} q_n^* = -q_{n-2}^* - (u_{n-1} + u_n)q_{n-1}^* - u_n^2 q_n^* - q_n q_{n-1}^* q_n^* - q_{n+1} q_n^{*2}, \\ \partial_{x_2} u_n = q_{n+2} q_n^* - q_n q_{n-2}^* + (u_{n+1} + u_n)(q_{n+1} q_n^*) - (u_{n-1} + u_n)(q_n q_{n-1}^*), \end{cases}$$

$$\begin{cases} \partial_{y_2} r_n = -v_n v_{n-1} r_{n-2} - v_n (r_{n-1} r_{n-1}^* + r_n r_n^*) r_{n-1}, \\ \partial_{y_2} r_n^* = v_{n+2} v_{n+1} r_{n+2}^* + v_{n+1} (r_n r_n^* + r_{n+1} r_{n+1}^*) r_{n+1}^*, \\ \partial_{y_2} v_n = r_n r_{n+1}^* v_{n+1} v_n^2 - v_n v_{n-1} r_{n-2} r_{n-1}^* v_{n-2} + v_n^2 (r_n^2 r_n^{*2} - r_{n-1}^2 r_{n-1}^{*2}). \end{cases} \quad (4.22)$$

## 5. ADDITIONAL BLOCK SYMMETRY OF TODA HIERARCHY

Here we will do a further reduction over the Lax operators (4.1) and (4.2) by letting them depend on the same time variables  $t_n$  with  $t_n = x_n = y_n$ . Then one can derive the constraint over the Lax operator  $\hat{\mathcal{L}}$  of the Toda hierarchy, i.e.

$$\hat{\mathcal{L}} = \mathcal{L} = \bar{\mathcal{L}} = L = \bar{L}. \quad (5.23)$$

Under this reduction, we denote reduced dressing operators  $\mathcal{P}, \bar{\mathcal{P}}$  as  $S, \bar{S}$  which have expansions of the form

$$\begin{aligned} S &= 1 + \omega_1(n) \Lambda^{-1} + \omega_2(n) \Lambda^{-2} + \cdots, \\ \bar{S} &= \bar{\omega}_0(n) + \bar{\omega}_1(n) \Lambda + \bar{\omega}_2(n) \Lambda^2 + \cdots. \end{aligned} \quad (5.24)$$

The inverse operators  $S^{-1}, \bar{S}^{-1}$  of operators  $S, \bar{S}$  have expansions of the form

$$\begin{aligned} S^{-1} &= 1 + \omega'_1(n) \Lambda^{-1} + \omega'_2(n) \Lambda^{-2} + \cdots, \\ \bar{S}^{-1} &= \bar{\omega}'_0(n) + \bar{\omega}'_1(n) \Lambda + \bar{\omega}'_2(n) \Lambda^2 + \cdots. \end{aligned} \quad (5.25)$$

The Lax operator  $\hat{\mathcal{L}}$  of the Toda hierarchy has the following expansions

$$\hat{\mathcal{L}} = \Lambda + U(n) + V(n) \Lambda^{-1}. \quad (5.26)$$

In fact the Lax operator  $\hat{\mathcal{L}}$  can also be equivalently defined by

$$\hat{\mathcal{L}} := S \circ \Lambda \circ S^{-1} = \bar{S} \circ \Lambda^{-1} \circ \bar{S}^{-1}. \quad (5.27)$$

In this section we will use a convenient notation on the operators  $\hat{B}_j$  defined as follows

$$\hat{B}_j := \frac{\hat{\mathcal{L}}^j}{j!}. \quad (5.28)$$

Now the known definition of the Toda hierarchy is as following.

**Definition 5.1.** *The Toda hierarchy is a hierarchy in which the dressing operators  $S, \bar{S}$  satisfy the following Sato equations [5]*

$$\partial_{t_j} S = -(\hat{B}_j)_- S, \quad \partial_{t_j} \bar{S} = (\hat{B}_j)_+ \bar{S}. \quad (5.29)$$

From the previous, one can derive the following well-known Lax equations of the Toda Hierarchy are as follows [5]

$$\partial_{t_j} \hat{\mathcal{L}} = [(\hat{B}_j)_+, \hat{\mathcal{L}}]. \quad (5.30)$$

We now put the constraint eq.(5.27) into a construction of the flows of additional symmetries which form the well-known Block algebra. With the dressing operators given in the eq.(5.27), we introduce Orlov-Schulman operators as following

$$M = S\Gamma S^{-1}, \quad \bar{M} = \bar{S}\bar{\Gamma}\bar{S}^{-1}, \quad (5.31)$$

$$\Gamma = n\Lambda^{-1} + \sum_{n \geq 0} (n+1)\Lambda^n t_n, \quad \bar{\Gamma} = -n\Lambda. \quad (5.32)$$

Then one can prove the Lax operator  $\hat{\mathcal{L}}$  and Orlov-Schulman operators  $M, \bar{M}$  satisfy the following theorem.

**Proposition 5.2.** *The Lax operator  $\hat{\mathcal{L}}$  and Orlov-Schulman operators  $M, \bar{M}$  of the Toda hierarchy satisfy the following*

$$[\hat{\mathcal{L}}, M] = 1, \quad [\hat{\mathcal{L}}, \bar{M}] = 1, \quad (5.33)$$

$$\partial_{t_n} M = [(B_n)_+, M], \quad \partial_{t_n} \bar{M} = [(B_n)_+, \bar{M}], \quad (5.34)$$

$$\frac{\partial M^m \hat{\mathcal{L}}^k}{\partial t_n} = [(B_n)_+, M^m \hat{\mathcal{L}}^k], \quad \frac{\partial \bar{M}^m \hat{\mathcal{L}}^k}{\partial t_n} = [(B_n)_+, \bar{M}^m \hat{\mathcal{L}}^k]. \quad (5.35)$$

*Proof.* One can prove the eq.(5.34) in this proposition by dressing the following two commutative Lie brackets

$$[\partial_{t_n} - \frac{\Lambda^{n+1}}{(n+1)!}, \Gamma] = 0, \quad [\partial_{t_n}, \bar{\Gamma}] = 0.$$

The other identities can be proved in a similar way. □

We now define the additional flows, and will then prove that they are additional symmetries of the Toda hierarchy. We introduce additional independent variables  $t_{m,l}$  and define the actions of the additional flows on the wave operators as

$$\frac{\partial S}{\partial t_{m,l}} = - \left( (M - \bar{M})^m \hat{\mathcal{L}}^l \right)_- S, \quad \frac{\partial \bar{S}}{\partial t_{m,l}} = \left( (M - \bar{M})^m \hat{\mathcal{L}}^l \right)_+ \bar{S}, \quad (5.36)$$

where  $m \geq 0, l \geq 0$ . By performing the derivative on  $\hat{\mathcal{L}}$  dressed by  $S$  and using the additional flow about  $S$  in (5.36), we get

$$\begin{aligned} (\partial_{t_{m,l}} \hat{\mathcal{L}}) &= (\partial_{t_{m,l}} S) \Lambda S^{-1} + S \Lambda (\partial_{t_{m,l}} S^{-1}) \\ &= -((M - \bar{M})^m \hat{\mathcal{L}}^l)_- S \Lambda S^{-1} - S \Lambda S^{-1} (\partial_{t_{m,l}} S) S^{-1} \\ &= -((M - \bar{M})^m \hat{\mathcal{L}}^l)_- \hat{\mathcal{L}} + \hat{\mathcal{L}} ((M - \bar{M})^m \hat{\mathcal{L}}^l)_- \\ &= -[((M - \bar{M})^m \hat{\mathcal{L}}^l)_-, \hat{\mathcal{L}}]. \end{aligned}$$

Similarly, we perform the derivative on  $\hat{\mathcal{L}}$  dressed by  $\bar{S}$  and use the additional flow about  $\bar{S}$  in (5.36) to get the following

$$(\partial_{t_{m,l}} \hat{\mathcal{L}}) = (\partial_{t_{m,l}} \bar{S}) \Lambda \bar{S}^{-1} + \bar{S} \Lambda (\partial_{t_{m,l}} \bar{S}^{-1})$$

$$\begin{aligned}
&= ((M - \bar{M})^m \hat{\mathcal{L}}^l)_+ \bar{S} \Lambda^{-1} \bar{S}^{-1} - \bar{S} \Lambda \bar{S}^{-1} (\partial_{t_{m,l}} \bar{S}) \bar{S}^{-1} \\
&= ((M - \bar{M})^m \hat{\mathcal{L}}^l)_+ \hat{\mathcal{L}} - \hat{\mathcal{L}} ((M - \bar{M})^m \hat{\mathcal{L}}^l)_+ \\
&= [((M - \bar{M})^m \hat{\mathcal{L}}^l)_+, \hat{\mathcal{L}}].
\end{aligned}$$

Because

$$[M - \bar{M}, \hat{\mathcal{L}}] = 0, \quad (5.37)$$

therefore one can further derive the following equation

$$\frac{\partial \hat{\mathcal{L}}}{\partial t_{m,l}} = [-((M - \bar{M})^m \hat{\mathcal{L}}^l)_-, \hat{\mathcal{L}}] = [((M - \bar{M})^m \hat{\mathcal{L}}^l)_+, \hat{\mathcal{L}}], \quad (5.38)$$

which gives the compatibility of additional flows of the Toda hierarchy with the reduction condition (5.27).

By the two propositions above, we can prove the additional flows  $\partial_{t_{m,l}}$  commute with the  $\partial_{t_n}$  flows of the Toda hierarchy, i.e.,

$$[\partial_{t_{m,l}}, \partial_{t_n}] \Phi = 0, \quad (5.39)$$

where  $\Phi$  can be  $S$ ,  $\bar{S}$  or  $\hat{\mathcal{L}}$ , and  $\partial_{t_{m,l}} = \frac{\partial}{\partial t_{m,l}}$ ,  $\partial_{t_n} = \frac{\partial}{\partial t_n}$ . The proof is just a direct calculation and standard. One can check the similar proof in [10], [18]-[20]. This commutative property means that additional flows are symmetries of the Toda hierarchy. Since they are symmetries, it is natural to consider the algebraic structures among these additional symmetries. So we obtain the following important theorem.

**Theorem 5.3.** *The additional flows  $\partial_{t_{m,l}}$  of the Toda hierarchy form a Block type Lie algebra as*

$$[\partial_{t_{m,l}}, \partial_{t_{n,k}}] = (km - nl) \partial_{t_{m+n-1, k+l-1}}, \quad (5.40)$$

which holds true in the sense of acting on  $S$ ,  $\bar{S}$  or  $\hat{\mathcal{L}}$  and  $m, n, l, k \geq 0$ .

*Proof.* By using (5.36), we get

$$\begin{aligned}
[\partial_{t_{m,l}}, \partial_{t_{n,k}}] S &= \partial_{t_{m,l}} (\partial_{t_{n,k}} S) - \partial_{t_{n,k}} (\partial_{t_{m,l}} S) \\
&= -\partial_{t_{m,l}} \left( ((M - \bar{M})^n \hat{\mathcal{L}}^k)_- S \right) + \partial_{t_{n,k}} \left( ((M - \bar{M})^m \hat{\mathcal{L}}^l)_- S \right) \\
&= -(\partial_{t_{m,l}} ((M - \bar{M})^n \hat{\mathcal{L}}^k)_- S) - ((M - \bar{M})^n \hat{\mathcal{L}}^k)_- (\partial_{t_{m,l}} S) \\
&\quad + (\partial_{t_{n,k}} ((M - \bar{M})^m \hat{\mathcal{L}}^l)_- S) + ((M - \bar{M})^m \hat{\mathcal{L}}^l)_- (\partial_{t_{n,k}} S).
\end{aligned}$$

Then by a tedious calculation, one can further get

$$\begin{aligned}
&[\partial_{t_{m,l}}, \partial_{t_{n,k}}] S \\
&= - \left[ \sum_{p=0}^{n-1} (M - \bar{M})^p (\partial_{t_{m,l}} (M - \bar{M})) (M - \bar{M})^{n-p-1} \hat{\mathcal{L}}^k + (M - \bar{M})^n (\partial_{t_{m,l}} \hat{\mathcal{L}}^k) \right]_- S \\
&\quad - ((M - \bar{M})^n \hat{\mathcal{L}}^k)_- (\partial_{t_{m,l}} S)
\end{aligned}$$

$$\begin{aligned}
 & + \left[ \sum_{p=0}^{m-1} (M - \bar{M})^p (\partial_{t_{n,k}}(M - \bar{M})) (M - \bar{M})^{m-p-1} \hat{\mathcal{L}}^l + (M - \bar{M})^m (\partial_{t_{n,k}} \hat{\mathcal{L}}^l) \right]_- S \\
 & + ((M - \bar{M})^m \hat{\mathcal{L}}^l)_- (\partial_{t_{n,k}} S) \\
 & = [(nl - km)(M - \bar{M})^{m+n-1} \hat{\mathcal{L}}^{k+l-1}]_- S \\
 & = (km - nl) \partial_{t_{m+n-1, k+l-1}} S.
 \end{aligned}$$

Similarly the same results on  $\bar{S}$  and  $\hat{\mathcal{L}}$  are as follows

$$\begin{aligned}
 [\partial_{t_{m,l}}, \partial_{t_{n,k}}] \bar{S} & = ((km - nl)(M - \bar{M})^{m+n-1} \hat{\mathcal{L}}^{k+l-1})_+ \bar{S} \\
 & = (km - nl) \partial_{t_{m+n-1, k+l-1}} \bar{S}, \\
 [\partial_{t_{m,l}}, \partial_{t_{n,k}}] \hat{\mathcal{L}} & = \partial_{t_{m,l}} (\partial_{t_{n,k}} \hat{\mathcal{L}}) - \partial_{t_{n,k}} (\partial_{t_{m,l}} \hat{\mathcal{L}}) \\
 & = [((nl - km)(M - \bar{M})^{m+n-1} \hat{\mathcal{L}}^{k+l-1})_-, \hat{\mathcal{L}}] \\
 & = (km - nl) \partial_{t_{m+n-1, k+l-1}} \hat{\mathcal{L}}.
 \end{aligned}$$

Now the theorem is proved.  $\square$

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